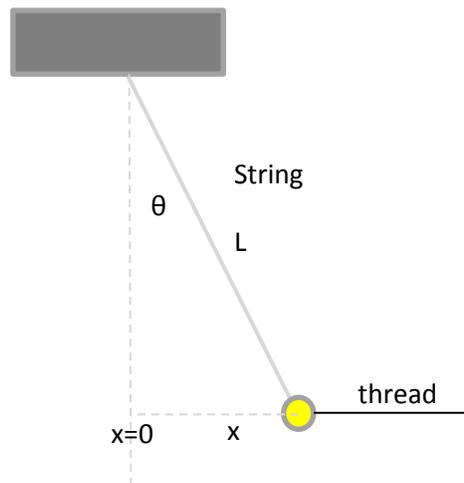
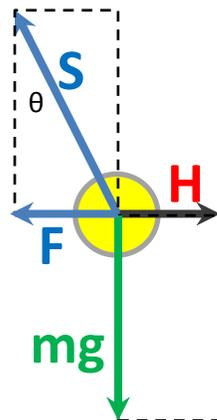


Acceleration of Gravity and Error from a Pendulum Experiment by R.S. LaFleur

1. The acceleration of gravity, g , on earth at sea level, is often quoted as $9.81 \text{ [m/sec}^2\text{]}$ or $981 \text{ [cm/sec}^2\text{]}$. Have you ever wondered how this is known? Your local g can be measured in a simple experiment of a swinging pendulum. However, all measurements have some error and we can compare the results of our experiment to the quoted value and see if we are within the margin of error.
2. The swinging pendulum is simple. We have a string of length L connecting a ball of constant mass, m , to a fixed point above it.
3. Initially the ball is in a static state, not moving, so the string is vertical, supporting the full weight of the ball and the horizontal position, x , is defined as $x=0$ (see the figure below).
4. The ball is deflected from $x=0$ by attaching a thread and pulling horizontally to some initial position. Now the ball is in a static state again, but this time we examine the forces that hold its position fixed.



5. The string makes an angle, θ , relative to the original vertical position so some of the string tension, S , is in the vertical direction and some is horizontal, F , opposed by the thread force, H . The force balance on the ball is shown below (F is in the negative x direction and H is positive).



6. In the static position, there is a vertical force balance and a horizontal force balance. The vertical force from gravity, mg , influences these forces through the string angle. For small angles, the tangent of the string angle is approximated as x/L

$$F = -mg \tan(\theta) \approx -mg \frac{x}{L}$$

7. When the thread is cut, the forces on the ball are from the string tension and the weight of the ball through gravity. Now with $H=0$, the horizontal force from the string, F , creates an acceleration in the horizontal direction. Newton's Second Law provides the relationship for the horizontal acceleration in terms of constant mass and horizontal force

$$ma = F$$

8. From experience, the pendulum swings and oscillates back and forth and this can be tracked along the horizontal direction with position x that is both $+$ and $-$ about $x=0$.
9. We can propose a model that follows a cosine wave of some unknown amplitude and frequency or period.

$$x = A \cos(2\pi ft)$$

10. This model also has its corresponding speed and acceleration by performing time derivatives.

$$v = -A (2\pi f) \sin(2\pi ft)$$

$$a = -A (2\pi f)^2 \cos(2\pi ft)$$

A frictionless assumption maintains the motion has constant amplitude over time.

11. Upon substitutions, we can solve for the acceleration of gravity in terms of the string length, L and the period of oscillation, T (note period is the inverse of frequency).

$$g = \left(\frac{2\pi}{T}\right)^2 L$$

This formula is valid only for small angles and frictionless motion.

12. Note that the initial amplitude does not participate in the solution of g , and therefore we can run different A values and we should obtain the same g . This also shows that the period of oscillation is the same for different A 's.
13. The experiment is designed to measure the string length, L [cm], and cycle period, T [sec]. Both have error from the real conditions of obtaining these experimental measurements. The measurement of string length is with a ruler and its error is derived from the scale resolution.

When experiments are run, the pendulum is deflected and then a timer started when the thread is cut or the ball is released. The timer is stopped after one or more cycles, each cycle ending when the ball returns momentarily to its starting point. The period is the amount of time for one cycle to complete and running more cycles before stopping the timer gives an average period. The average period may be a way to minimize the effect of judging starting and stopping positions and operating the timer by hand (stopwatch). More sophisticated timing triggers can be used to remove the human-judgment&stopwatch effect. The end result is the period, T , and its error.

14. With the two measurements completed, the error can be propagated to the acceleration of gravity. For example, the following results were obtained in the experiment

Measurements	Center	±Error	Units
Length, L	92.95	0.1	[cm]
Period, T	1.936	0.004	[sec]

15. The ‘differential method’ uses partial differentials to generate an error propagation formula. In this problem, the convenient way to apply the differential method is using relative error. This works as long as the errors are small and there are no divide-by-zeros when relative error is formulated.

The generation of the error propagation formula is a typical assignment in beginning physics courses and dovetails with the calculus curriculum. This takes work and time we can call ‘derivation time.’

The answers, using the differential method are the following two formulae, one for the ‘center’ and one for the ‘error.’

$$g = \left(\frac{2\pi}{T}\right)^2 L$$

$$eg = g \sqrt{\left(\frac{eL}{L}\right)^2 + 4\left(\frac{eT}{T}\right)^2}$$

This assumes the irrational number, 2π , is exact or ‘has zero error.’ The calculation procedure is sequential (not simultaneous) as the center, g , is calculated first followed by the error, eg , calculation. Consult the accompanying spreadsheet for the ‘Conventional Procedure’ worksheet that details 29 calculation steps and offers cells to record your answers. You can use a conventional scientific calculator, such as a TI-89, as the description of steps will be similar. Note that step 9 requires you to write down the answer for g .

You should time yourself to obtain a ‘scientific calculator time.’ This time, added to the ‘derivation time’ gives an overall measure of performance.

16. The CertainError calculator automates the error propagation and makes the derivation of error propagation formula unnecessary. For this problem, the formula has two independent measurements and the ‘derivation time’ is zero. The four numbers shown in the above table are input into the CertainError calculator and basic arithmetic is performed much like the ‘center’ calculation on the scientific calculator. Each number has two parts (tilde)

$$\tilde{g} = \left(\frac{2\pi}{\tilde{T}}\right)^2 \tilde{L}$$

To compare answers, method used on the CertainError should be set to 'differential.' This results in a much shorter and faster procedure. Consult the accompanying spreadsheet for the 'CertainError app procedure' worksheet that details 15 calculation steps. Unlike the scientific calculator, the CertainError calculator does not require any 'writing down' of intermediate answers.

You should time yourself to obtain a 'CertainError calculator time.' Since 'derivation time' is zero, this gives an overall measure of performance.

17. One comparison of the scientific calculator and CertainError calculator is on the resulting answers for the gravity and its error.

g Calculation	Center	±Error	Units
Conventional Scientific			cm/sec ²
CertainError			cm/sec ²

The answers for the gravity and its error should be nearly the same. These answers are also compared to the quoted value of the acceleration of gravity of 981 [cm/sec²]. Are the center answers closer than the error bar?

18. Another comparison is on the calculation times. This is a head-to-head comparison of the ability to crunch numbers rather than taxing the comparison with the 'derivation time.' It is expected that the CertainError calculator will be much faster as the procedure has less steps and the CertainError calculator performs the calculation using an internal algorithm that the scientific calculator does not have. You are paying for this advantage and time savings.

See the accompanying video of a student performing both calculations for this swinging pendulum problem. The results show that the CertainError Calculator is over twice as fast. This advantage together with the fact that you do not spend time (and frustration) deriving error formula clearly shows the benefits of the CertainError calculator. This benefit has been shown for this one small problem. Once you have the CertainError calculator, the advantage can be achieved on any problem you do, putting you on a different plane of operation.